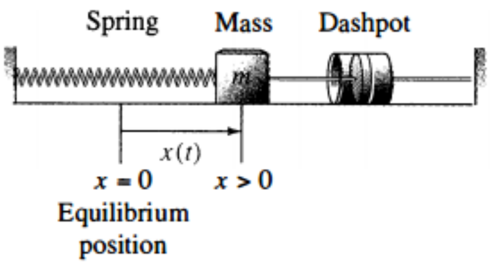
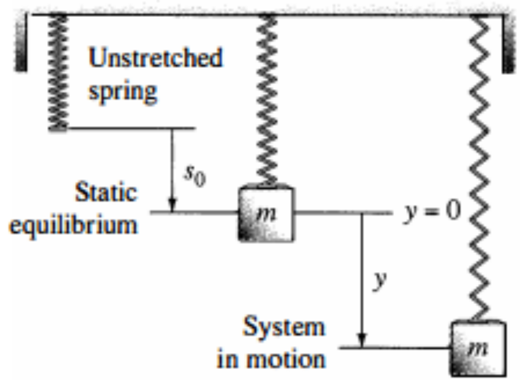
Solutions

3.4: Mechanical Vibrations



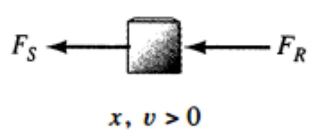


The weight of the mass stretches the spring

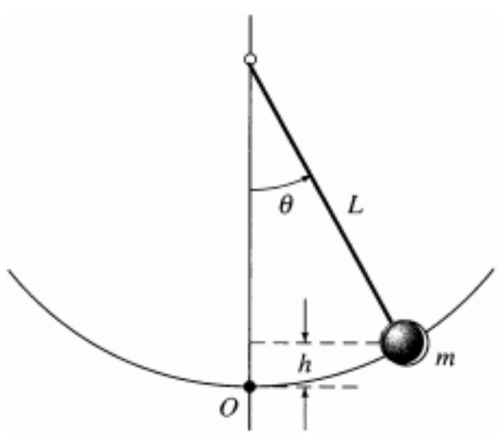
$S_0 = \frac{mg}{k}$ units.

Forced Motion $(F(t) \neq 0)$

This will be covered in later sections.



Free Motion $(F(t) = 0)$



Another example of a mechanical system is given by the **simple pendulum**. In this example we have a mass swinging at the end of a string of length L. We will study the motion of the mass through the counter-clockwise angle $\theta = \theta(t)$. Applying the law of the *conservation of mechanical energy*, which states that the sum of the kinetic and potential remains constant, we will obtain our differential equation.

Using the arc length formula for distance, we will get the kinetic energy is given by $T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$. Its potential energy is given by $V = mg \cdot L(1 - \cos\theta)$. We obtain the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0. \qquad (1)$$

Assuming θ never exceeds $\frac{\pi}{12}$ radians $= 15°$, we use the fact that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ so that $\sin\theta \approx \theta$.

Then everything follows as in next page.

We add the term $c\theta'$ to compensate for friction.

The diagrams on the previous page illustrate the main topics of this section. We wish to use linear differential equations to study the motion of a mass attached to a spring. We see in both pictures the **equilibrium position**, which is the resting position of the mechanical system.

We will take $F_S = -kx$ to be the restorative force of the spring to its equilibrium position. k is known as the **spring constant**. We will take $F_R = -cv$ to be something like a shock absorber. c is known as the **damping constant**. Sometimes we will consider including also an **external force** $F_E = F(t)$. In total, the force acting on the mass is given by $F = F_S + F_R + F_E$. Using Newton's Second Law of Motion, we get the differential equation

$$mx'' + cx' + kx = F(t). \tag{2}$$

The case of free, undamped motion is called **simple harmonic motion**.

Damped Motion if $c \neq 0$. Think of a shock absorber

Undamped Motion when $c = 0$.

Undamped Free Motion First: $F(t) = 0 = c$.

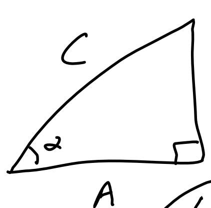
$$mx'' + kx = 0 \Rightarrow x'' + \frac{k}{m}x = 0 \Rightarrow x'' + \omega_0^2 x = 0,$$

where $\omega_0 = \sqrt{k/m}$ is the circular frequency.

Solve the characteristic eqn to get

$$(\text{A}) \quad x(t) = A\cos\omega_0 t + B\sin\omega_0 t.$$

The Amplitude $C = \sqrt{A^2 + B^2}$ tells us how far the mass moves.



The Phase angle α is given by

$$\alpha = \begin{cases} \tan^{-1} B/A, & A, B > 0 \text{ (1st Quadrant)} \\ \pi + \tan^{-1} B/A, & A < 0 \text{ (2nd \& 3rd Quadrants)} \\ 2\pi + \tan^{-1} B/A, & A > 0, B < 0 \text{ (4th Quadrant)}. \end{cases}$$

We use trig to simplify

(A) to $x(t) = C\cos(\omega_0 t - \alpha)$

$= C\cos(\omega_0(t - \delta))$, where $\delta = \frac{\alpha}{\omega_0}$ is the time lag.

Finally the period is given by $\frac{2\pi}{\omega_0}$ and frequency is $\frac{\omega_0}{2\pi}$ Hz which represents cycles/s.

Example 1. A body with mass $m = \frac{1}{2}$ is attached to a spring that is stretched 2 m by a force of 100 N. It is set in motion with initial position $x_0 = 1$ m and initial velocity $v_0 = -5$ m/s. Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Spring constant $\equiv k = \frac{100\text{N}}{2\text{m}} = 50 \text{ N/m}$.

This gives $\frac{1}{2}x'' + 50x = 0 \Rightarrow x'' + 100x = 0$ so that circular frequency $= \omega_0 = \sqrt{100} = 10$ and therefore

$$x(t) = A\cos 10t + B\sin 10t.$$

Given initial conditions, we find

$$x(0) = 1 = A \quad \text{and} \quad x'(0) = -5 = 10B \Rightarrow B = -\frac{1}{2}.$$

Therefore amplitude $\equiv C = \sqrt{1 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{5}$ m,

the phase angle $\equiv \alpha = 2\pi + \tan^{-1}(-1/2) \approx 5.8195$,

and the time lag $\equiv \delta = \frac{\omega_0}{\alpha} \approx 0.582$

We can then rewrite

$$x(t) = \frac{\sqrt{5}}{2}\cos(10t - 5.8195) = \frac{\sqrt{5}}{2}\cos(10(t - 0.582)).$$

Lastly, we have period $\equiv T = \frac{2\pi}{10} \approx 0.314$

and frequency $\equiv \nu = \frac{1}{T} = \frac{10}{\pi}$ Hz.

Next we consider free damped motion. Thus $c \neq 0$ and $F(t)=0$.

$mx'' + cx' + Kx = 0$

$x'' + 2px' + w_0^2 x = 0$,

where

$w_0 = \sqrt{K/m}$ and $p = \frac{c}{2m} > 0$.

We consider p for different damping regimes

So $r_1, r_2 = -p \pm (p^2 - w_0^2)^{1/2}$. $c_{cr} = \sqrt{4Km}$

Over Damped: $c > c_{cr}$, $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Critically Damped: $c = c_{cr}$, $x(t) = (c_1 + c_2 t) e^{-pt}$

Under Damped: $c < c_{cr}$, $x(t) = e^{-pt}(A\cos w_1 t + B\sin w_1 t)$

where $w_1 = \sqrt{w_0^2 - p^2}$ is circular freq.

Example 2. The mass and spring of Example 1 are now attached also to a dashpot that provides 1 N of resistance for each meter per second of velocity. The initial position and velocity are the same as Example 1. Find the position function of the mass and the time it takes for it to pass four times through the initial position.

$m = \frac{1}{2}$, $K = 50$, $c = 1$.

So $\frac{1}{2}x'' + x' + 50x = 0 \Rightarrow x'' + 2x' + 100x = 0$.

Then $c_{cr} = \sqrt{100} = 10$. Under damped w/ roots $-1 \pm \sqrt{99}i$

So $x(t) = e^{-t}(A\cos\sqrt{99}t + B\sin\sqrt{99}t)$. Solve for $A = 1$, $B = -4/\sqrt{99}$.

Thus $x(t) = e^{-t}\left(\cos\sqrt{99}t - \frac{4}{\sqrt{99}}\sin\sqrt{99}t\right)$

$\approx \sqrt{\frac{115}{99}}\, e^{-t}\cos(\sqrt{99}t - 5.9)$ so that $t_4 \approx 1.0667$ s.

Homework. 1-17 (odd), 24-26 (all)